

ON TORAL EIGENFUNCTIONS AND THE RANDOM WAVE MODEL

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ABSTRACT. The purpose of this Note is to provide a deterministic implementation of the random wave model for the number of nodal domains in the context of the two-dimensional torus. The approach is based on recent work due to Nazarov and Sodin and arithmetical properties of lattice points on circles.

1. INTRODUCTION

This Note originates from the work of Nazarov and Sodin ([N-S] and [S]) on the behavior of the number of nodal domains of random eigenfunctions at high energy. It was shown in [N-S] that the number N_E of a random eigenfunction of S^2 of eigenvalue E obeys the so-called random wave model (RWM) for large E and, with large probability, the ratio $4\pi \frac{N_E}{E}$ is close to a constant $\sigma > 0$. According to the Bogomolny-Schmit [B-S1], [B-S2] prediction, this number σ can be computed based on a bond percolation model leading to a conjectured value

$$\sigma = \frac{3\sqrt{3} - 5}{\pi} \approx 0.0624... \quad (1.1)$$

While the [N-S] work establishes in particular the positivity of σ , it does not shed any light on its actual value. Note that (1.1) is considerably smaller than the general (deterministic) upper bound provided by Pleijel's inequality

$$\limsup_{n \rightarrow \infty} \frac{N}{n} \leq \left(\frac{2}{j}\right)^2 = 0.691.. \quad (1.2)$$

with j the smallest positive zero of the Bessel function J_0 and $n \asymp \frac{E}{4\pi}$ the wave number (see also [B] for a small improvement).

Better upper bounds on σ may be obtained by evaluation of certain geometric parameters using Kac-Rice type arguments. It was shown in particular in [K] that $\sigma \leq \frac{1}{\sqrt{2\pi}} = 0.225...$ by computation of the expected number of horizontal tangencies to the nodal set. The same bound may be gotten from its expected total curvature (cf. [Ber]).

In what follows, we do not intend to study further the RWM or the Bogomolny-Schmit heuristics. Rather, we are interested in a deterministic implementation of the RWM in certain situations. The idea is very simple. Assume $-\Delta f = Ef$

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an eigenfunction for large E . Fixing some base point $x \in M$ in the manifold E , we are considering restrictions f_x of f to neighborhoods of x of the order $O(\frac{1}{\sqrt{E}})$ (in fact $\frac{R}{\sqrt{E}}$ with R slowly growing to infinity with E). In certain instances, one may then show that the ensemble $(f_x)_{x \in M}$ resembles that of a Gaussian random wave function. It turns out that for $M = \mathbb{T}^2$ the 2-dimensional flat torus and eigenfunctions

$$f(x) = \sum_{|\xi|^2=E} a_\xi e(x \cdot \xi) \quad (e(a) = e^{2\pi i a}) \quad (1.3)$$

($a_{-\xi} = \bar{a}_\xi$) where $\mathcal{E}_E = \{\xi \in \mathbb{Z}^2; |\xi|^2 = E\}$ satisfies suitable arithmetical assumptions and $\sum_{|\xi|^2=E} |a_\xi|^2 \delta_{\xi/\sqrt{E}}$ becomes well-distributed on the unit circle, this idea may be worked out rather easily. On the arithmetic side, we rely on angular equidistribution results [E-H] (see also [F-K-W] and related references) and also the recent work [B-B] on additive relations in the set \mathcal{E}_E . Naturally, one runs into stability problems for the number of nodal domains when perturbing slightly the eigenfunctions, but these analytical issues have been already completely addressed as part of the remarkable work of Sodin and Nazarov. In particular, extensive use is made from the results in [S].

Recall also that from a result due to A. Stern [St] (see also [L]), there is no nontrivial lower bound on the number of nodal domains for $E \rightarrow \infty$, which may equal two. Thus for eigenfunctions (1.3), some further assumptions are needed. Possibly, the equidistribution of the measures $\sum_{|\xi|^2=E} |a_\xi|^2 \delta_{\xi/\sqrt{E}}$ on S^1 may suffice, but we are only able to establish this in certain cases (for instance assuming E has a bounded number of prime factors and also in a statistical sense, i.e. for ‘most’ E).

Beyond the arithmetical input and the results from [S], our analysis is essentially straightforward. No effort has been made to obtain quantitatively more refined results. A more general outlook on the approach is discussed in the last section.

Let us return to our model \mathbb{T}^2 and be more specific.

Assume $E \in \mathbb{Z}_+$ a large odd integer which is a sum of 2 squares; we assume moreover E of the form

$$E = \prod p_\alpha^{e_\alpha} \quad (e_\alpha \geq 1) \quad (1.4)$$

where its prime factors $p_\alpha \equiv 1 \pmod{4}$. Denote

$$\mathcal{E} = \mathcal{E}_E = \{\xi = (\xi_1, \xi_2) \in \mathbb{Z}^2; \xi_1^2 + \xi_2^2 = E\}. \quad (1.5)$$

Identifying (ξ_1, ξ_2) with the Gaussian integer $\xi_1 + i\xi_2 \in \mathbb{Z} + i\mathbb{Z}$ and denoting $p_\alpha = \pi_\alpha \bar{\pi}_\alpha$ the factorization of p_α in Gaussian primes, the set \mathcal{E} is obtained as

$$\left\{ \prod_\alpha \pi_\alpha^j \bar{\pi}_\alpha^{e_\alpha-j}; 0 \leq j \leq e_\alpha \right\} \quad (1.6)$$

up to multiplication by ± 1 and $\pm i$. In particular

$$|\mathcal{E}| = 4 \cdot \prod (1 + e_\alpha) = W. \quad (1.7)$$

Writing $E = \lambda^2$ and

$$\pi_\alpha = |\pi_\alpha| e^{i\theta_\alpha} \quad (1.8)$$

we obtain

$$\xi = \lambda e^{i\psi} \text{ for } \xi \in \mathcal{E}_E \quad (1.9)$$

with angles

$$\psi = \sum_{\alpha} (2j_\alpha - e_\alpha) \theta_\alpha \text{ and } 0 \leq j_\alpha \leq e_\alpha \quad (1.10)$$

up to multiples of $\frac{\pi}{2}$.

The eigenfunctions of the Laplacian $-\Delta$ on \mathbb{T}^2 with eigenvalue E are obtained as trigonometric polynomials

$$f = \sum_{\xi \in \mathcal{E}_E} a_\xi e(x \cdot \xi). \quad (1.11)$$

Let us consider for simplicity the eigenfunction

$$\sum_{\xi \in \mathcal{E}_E} e(x \cdot \xi). \quad (1.12)$$

Our considerations in the remainder of the paper carry over verbatim to the situation (1.11) with $|a_\xi|, \xi \in \mathcal{E}$ equal and more general statements with arbitrary coefficients $(a_\xi)_{\xi \in \mathcal{E}}$ will be discussed later.

Our aim is to show that under suitable assumptions on $E \rightarrow \infty$, the number of nodal domains of (1.12) obeys the RWM. These assumptions are of arithmetical nature and may be loosely formulated as follows

- (D) The points $\{\lambda^{-1}\xi; \xi \in \mathcal{E}_E\}$ become equidistributed on the unit circle for $E \rightarrow \infty$.
- (I) There are not too many non-trivial additive relations among the elements of \mathcal{E} .

While we only need (D) without further quantification, a more precise form of (I) will be required (See Definition 1 and Proposition 1). Properties (D) and (I) may be addressed by classical results in number theory. By (1.10), (D) relates to angular distribution of Gaussian primes and we will refer to the results from [E-H]. A powerful tool to deduce bounds on the number of additive relations is provided by [E-S-S] on unit equations

$$a_1 \xi_1 + \cdots + a_\ell \xi_\ell = 1 \quad (1.13)$$

with ξ_1, \dots, ξ_ℓ taken in a multiplicative subgroup G of \mathbb{C}^* of bounded rank (though the available results require some further assumption on the number of prime factors of E to be applicable to our problem). Alternatively, one may use the ‘statistical’

results on additive relations proven in [B-B] to treat the case of ‘typical’ E . Precise statements will be given in section 4 (Theorems 2, 3, 4).

2. LOCAL ANALYSIS OF THE EIGENFUNCTION

Let \mathbb{T}^2 be equipped with normalized measure and let

$$f(x) = \frac{1}{\sqrt{W}} \sum_{\xi \in \mathcal{E}} e(x \cdot \xi) \quad (2.1)$$

with $\mathcal{E} = \mathcal{E}_E$, $E = \lambda^2$ and $W = |\mathcal{E}|$. We always assume $W \rightarrow \infty$ with $E \rightarrow \infty$.

In what follows, we will need several parameters, chosen in a particular order, that will be viewed as $O(1)$ for fixed E and eventually will tend to infinity with $E \rightarrow \infty$ at sufficiently slow rate.

Let $1 \ll K = o(W)$ be a first large parameter and subdivide λS^1 in arcs of size $\frac{\lambda}{K}$, leading to a corresponding partition

$$\mathcal{E} = \bigcup_{k=1}^K \mathcal{E}^{(k)}. \quad (2.2)$$

More specifically, we subdivide the first quadrant of λS^1 and partition the other regions by reflection and symmetry. According to (D), assume that

$$\left(\frac{1}{K} - \varepsilon_1\right)W < |\mathcal{E}^{(k)}| < \left(\frac{1}{K} + \varepsilon_1\right)W \quad (2.3)$$

for each $k = 1, \dots, K$, where $\varepsilon_1 = \varepsilon_1(K)$. Choose a point $\xi^{(k)} \in \mathcal{E}^{(k)}$, letting $\xi^{(k')} = -\xi^{(k)}$ if $\mathcal{E}^{(k')} = -\mathcal{E}^{(k)}$.

Let $R \gg 1$ be another parameter and denote

$$\zeta^{(k)} = \frac{R}{\lambda} \xi^{(k)}. \quad (2.4)$$

Hence $|\zeta^{(k)}| = R$. Fixing $x \in \mathbb{T}^2$, translate x by $\frac{R}{\lambda}y$ with $y = (y_1, y_2) \in [-\frac{1}{2}, \frac{1}{2}]^2$ and write

$$F_x(y) = f\left(x + \frac{R}{\lambda}y\right) = \frac{1}{\sqrt{K}} \sum_{k=1}^K f_k(x, y) e(\zeta^{(k)} \cdot y) \quad (2.5)$$

with

$$f_k(x, y) = \sqrt{\frac{K}{W}} \sum_{\xi \in \mathcal{E}^{(k)}} e\left(\xi \cdot x + \left(\frac{R\xi}{\lambda} - \zeta^{(k)}\right) \cdot y\right). \quad (2.6)$$

Denote further

$$\varphi(y) = \varphi_x(y) = \frac{1}{\sqrt{K}} \sum_{k=1}^K c_k(x) e(\zeta^{(k)} \cdot y) \quad (2.7)$$

with

$$c_k(x) = f_k(x, 0). \quad (2.8)$$

Our next goal is to show the following

- (i) For most x , φ_x is a perturbation of F_x considered as function of y ,

- (ii) The random vector $\{c_k(x)\}_{1 \leq k \leq K}$ with x ranging in \mathbb{T}^2 has approximately the same distribution as the Gaussian vector $\{g_k\}_{1 \leq k \leq K}$, with g_1, \dots, g_K IID normalized complex Gaussians, subject to the reality condition $g_{k'} = \bar{g}_k$ for $\zeta^{(k')} = -\zeta^{(k)}$. At this point, we will then be able to rely on the results from [S].

Note that in (ii), we should see K as fixed and the distributional approximation sufficient in order for the relevant Gaussian estimates from [S] to carry over.

Lemma 1. *For any fixed $s \geq 1$,*

$$\|F_x - \varphi_x\|_{L_x^2 C_y^s} < R^{C_s} K^{-1}. \quad (2.9)$$

Proof. Since from standard Sobolev estimates, we may bound the C^s -norm by the H^{s+2} -norm, it suffices to estimate

$$\left\| f\left(x + \frac{R}{\lambda} y\right) - \varphi_x \right\|_{L_x^2 H_y^s} \leq C \frac{R^s}{\sqrt{W}} \left(\sum_{k=1}^K \sum_{\xi \in \mathcal{E}^{(k)}} \left| \frac{R}{\lambda} \xi - \zeta^{(k)} \right|^2 \right)^{\frac{1}{2}} < C \frac{R^{s+1}}{K}$$

by (2.5)–(2.8). \square

It follows from (2.9) that after fixing R , we may ensure, taking K sufficiently large, that

$$\|F_x - \varphi_x\|_{C^s} = o(1) \quad (2.10)$$

for most $x \in \mathbb{T}^2$.

We now turn our attention to the joint distribution of the vector $\{C_k(x); 1 \leq k \leq K\}$,

$$C_k(x) = \sqrt{\frac{K}{W}} \sum_{\xi \in \mathcal{E}^{(k)}} e(\xi \cdot x) \quad (2.11)$$

when x ranges in \mathbb{T}^2 .

Switching notation a bit, it will be convenient to replace K by $2K$ and enumerate $\mathcal{E}^{(1)}, \dots, \mathcal{E}^{(K)}, \mathcal{E}^{(-1)}, \dots, \mathcal{E}^{(-K)}$ with $\mathcal{E}^{(-k)} = -\mathcal{E}^{(k)}$. Obviously $c_{-k} = \bar{c}_k$.

We specify assumption (I) as follows.

Definition 1. *Fix $0 < \gamma < \frac{1}{2}$ and some $B \in \mathbb{Z}_+$. We say that \mathcal{E} satisfies property $I(\gamma, B)$ if for $2 < \ell \leq B$, the number of non-degenerate additive relations of the form*

$$\xi_1 + \dots + \xi_\ell = 0 \quad (2.12)$$

among elements $\xi_1, \dots, \xi_\ell \in \mathcal{E}$ is at most $N^{\gamma\ell}$. By ‘non-degenerate’, we mean that in (2.12) there is no proper vanishing sub-sum.

There are various ways to select energies E for which above independence property holds and this will be addressed in a later section.

Definition 2. Let $\varepsilon > 0$ be a small parameter. Say that the random vector (c_1, \dots, c_k) where the c_j are \mathbb{C} -valued functions, $\|c_j\|_2 \sim 1$, is ε -Gaussian, provided for any (possibly unbounded) intervals $I_1, J_1, \dots, I_K, J_K \subset \mathbb{R}$, we have

$$\left| \text{mes}[c_1 \in I_1 \times J_1, \dots, c_k \in I_K \times J_K] - \frac{1}{(2\pi)^K} \int_{I_1 \times J_1 \times \dots \times I_K \times J_K} e^{-\frac{1}{2}(x_1^2 + y_1^2 + \dots + x_K^2 + y_K^2)} dx dy \right| < \varepsilon. \quad (2.13)$$

Choosing ε sufficiently small (in particular depending on K), (2.13) will enable to approximate for ‘nice’ open sets $\Omega \subset \mathbb{C}^K$, $\text{mes}[(c_1, \dots, c_K) \in \Omega]$ by the corresponding Gaussian measure. We prove

Lemma 2. Given $\varepsilon > 0$, there is $B = B(K, \varepsilon)$ such that if \mathcal{E} satisfies $I(\gamma, B)$, then the vector function (c_1, \dots, c_K) on \mathbb{T}^2 as defined above, is ε -Gaussian.

Proof. Well-known arguments reduce the problem to evaluating moments

$$\int_{\mathbb{T}^2} c_1^{r_1} \bar{c}_1^{s_1} \dots c_K^{r_K} \bar{c}_K^{s_K} \quad (2.14)$$

with $r_1, s_1, \dots, r_K, s_K \in \mathbb{Z}_+ \cup \{0\}$ and

$$r_1 + s_1 + \dots + r_K + s_K < B_1(K, \varepsilon).$$

Recall that the procedure consists indeed in evaluating the characteristic function

$$\int_{\mathbb{T}^2} e^{[\alpha_1 \text{Re } c_1 + \beta_1 \text{Im } c_1 + \dots + \alpha_K \text{Re } c_K + \beta_K \text{Im } c_K]} dx \quad (2.15)$$

with $\alpha_1, \beta_1, \dots, \alpha_K, \beta_K \in \mathbb{R}$, subject to some bound $B_2(k, \varepsilon)$. Those arise by suitable truncations of the Fourier transform of intervals. Then, imposing some bound on $|c_1|, \dots, |c_K|$, Taylor expansion of the exponentials in (2.15) leads to the expressions (2.14).

Substituting (2.11) in (2.14) gives

$$\left(\frac{K}{W} \right)^{\frac{1}{2}(r_1 + \dots + r_K + s_1 + \dots + s_K)}. \quad (2.16)$$

where (2.16) stands for the number of relations

$$\xi_{1,1} + \dots + \xi_{1,r_1} - \xi'_{1,1} - \dots - \xi'_{1,s_1} + \dots + \xi_{K,1} + \dots + \xi_{K,r_K} - \xi'_{K,1} - \dots - \xi'_{K,s_K} = 0 \quad (2.17)$$

with $\xi_{1,1}, \dots, \xi_{1,r_1}$ and $\xi'_{1,1}, \dots, \xi'_{1,s_1} \in \mathcal{E}^{(1)}, \dots$

Trivial solutions to (2.17) are those for which the multi-sets (i.e. taking into account multiplicities)

$$\{\xi_{1,1}, \dots, \xi_{1,r_1}, \dots, \xi_{K,1}, \dots, \xi_{K,r_K}\} \quad (2.18)$$

and

$$\{\xi'_{1,1}, \dots, \xi'_{1,s_1}, \dots, \xi'_{K,1}, \dots, \xi'_{K,s_K}\}$$

coincide. Of course, to have a trivial solution requires

$$r_1 = s_1, \dots, r_K = s_K. \quad (2.19)$$

Otherwise, we call the solution non-trivial.

Consider first the contribution of trivial solutions, assuming (2.19).

Denote $\Omega = \mathbb{T}^W$ and define $\tilde{c}_1, \dots, \tilde{c}_K$ on \mathbb{T}^K by

$$\tilde{c}_k(\Psi) = \sqrt{\frac{K}{W}} \sum_{\xi \in \mathcal{E}^{(k)}} e(\psi_\xi) \text{ with } \Psi = (\psi_\xi)_{\xi \in \mathcal{E}}. \quad (2.20)$$

Recalling (2.3) and taking into account the central limit theorem, the distribution of $(\tilde{c}_1, \dots, \tilde{c}_K)$ is approximatively Gaussian. The trivial solutions to (2.17) contribute for

$$\int_{\Omega} |\tilde{c}_1|^{2r_1} \dots |\tilde{c}_K|^{2r_K} \cong \int |g_1|^{2r_1} \dots |g_K|^{2r_K}. \quad (2.21)$$

Consider next the contribution of non-trivial relations, which will be evaluated using our arithmetical assumption. Their number is obviously bounded by the number of nontrivial relations

$$\xi_1 + \dots + \xi_\ell = 0 \text{ with } \ell = r_1 + \dots + r_K + s_1 + \dots + s_K \quad (2.22)$$

in elements ξ from \mathcal{E} . Partitioning (20) in minimal vanishing sub-sums, at least one of these relations will be non-trivial and therefore of length $\ell' \geq 3$. Property $I(\gamma, B)$, $B \geq \ell$, clearly implies the following bound

$$C(\ell) \sum_{2\nu \leq \ell-3} W^\nu W^{\gamma(\ell-2\nu)} < C(\ell) W^{\frac{\ell}{2}} W^{-3(\frac{1}{2}-\gamma)}. \quad (2.23)$$

Multiplying with $(\frac{K}{W})^{\frac{\ell}{2}}$, the resulting contribution of the non-trivial relations (2.17) in (2.14) is therefore at most

$$B_3(K, \varepsilon) W^{-(\frac{1}{2}-\gamma)} \quad (2.24)$$

which can be made arbitrarily small for W large enough.

This proves Lemma 2. \square

3. THE NUMBER OF NODAL DOMAINS

Consider the eigenfunction (2.1) on \mathbb{T}^2 .

From general theory, the total length of the zero set $Z(f) = \{x \in \mathbb{T}^2; f(x) = 0\}$ is $O(\lambda)$ while each nodal domain has area at least $O(\lambda^{-2})$. In particular, it follows that the number of nodal domains of diameter at least $\varepsilon_2^{-1}\lambda^{-1}$ is at most $O(\varepsilon_2\lambda^2)$. Here ε_2 is a small fixed constant.

Choosing R sufficiently large, it clearly follows from the preceding that

$$N_f = \frac{\lambda^2}{R^2} \int_{\mathbb{T}^2} N_f\left(x, \frac{R}{\lambda}\right) dx + O\left(\varepsilon_2\lambda^2 + \frac{\lambda^2}{R\varepsilon_2}\right) \quad (3.1)$$

where N_f is the number of nodal domains of f and $N_f(x, \rho)$ the number of nodal domains contained in the open box $x + (]-\frac{\rho}{2}, \frac{\rho}{2}[\times]-\frac{\rho}{2}, \frac{\rho}{2}[)$.

Using our notation (2.5), the first term on the rhs of (3.1) equals

$$\frac{\lambda^2}{R^2} \int_{\mathbb{T}^2} N_{F_x} dx \quad (3.2)$$

with N_F the number of components of $Z(F)$ contained in $] -\frac{1}{2}, \frac{1}{2}[\times] -\frac{1}{2}, \frac{1}{2}[$. Note also that

$$N_{F_x} = N_f\left(x, \frac{R}{\lambda}\right) < O(R^2). \quad (3.3)$$

Let φ be defined by (2.7). According to Lemma 1

$$\int \|F_x - \varphi_x\|_{C^1} dx < R^C K^{-1}. \quad (3.4)$$

Hence, fixing another parameter $\varepsilon_3 > 0$, it follows that

$$\|F_x - \varphi_x\|_{C^1} < \varepsilon_3 \quad (3.5)$$

except for x in a subset $V \subset \mathbb{T}^2$ of measure at most $\varepsilon_3^{-1} R^C K^{-1} < \varepsilon_2$, taking K sufficiently large. Since then, by (3.3)

$$\frac{\lambda^2}{R^2} \int_V N_{F_x} dx < O(\lambda^2 |V|) < \varepsilon_2 \lambda^2$$

we may replace (3.2), up to $O(\varepsilon_2 \lambda^2)$, by

$$\frac{\lambda^2}{R^2} \int_{\mathbb{T}^2 \setminus V} N_{\varphi_x + \psi_x} dx \quad (3.6)$$

where $\|\psi_x\|_{C^1} < \varepsilon_3$.

For $x \in V$, set $\psi_x = 0$. Since the function φ_x on \mathbb{R}^2 satisfies $-\Delta \varphi_x = R^2 \varphi_x$, it follows again from the Faber-Krahn inequality that each nodal domain of φ_x is of area at least $O(\frac{1}{R^2})$ and hence $N_{\varphi_x} < O(R^2)$. Thus

$$\frac{\lambda^2}{R^2} \int_V N_{\varphi_x} dx < O(\lambda^2 |V|) < \varepsilon_2 \lambda^2$$

and in (3.6), the integral may be extended to \mathbb{T}^2 . Consequently, we obtain

$$\frac{\lambda^2}{R^2} \int_{\mathbb{T}^2} N_{\varphi_x + \psi_x} dx. \quad (3.7)$$

The next step consists in invoking Lemma 2, which asserts that for W sufficiently large, the ensemble $(\varphi_x)_{x \in \mathbb{T}^2}$ has approximately the same distribution as the Gaussian random function

$$\Phi_\omega = \frac{1}{\sqrt{K}} \sum_{k=1}^K g_k(\omega) e(\zeta^{(k)} \cdot y) \quad (3.8)$$

with $\{g_k\}$ IID normalized complex Gaussians subject to the condition $g_{k'} = \bar{g}_k$ for $\zeta^{(k')} = -\zeta^{(k)}$.

We claim that by choosing ε small enough in Lemma 2, one can replace (3.7) by

$$\frac{\lambda^2}{R^2} \int N_{\Phi_\omega + \Psi_\omega} d\omega \quad (3.9)$$

where Ψ_ω is some perturbing function, satisfying

$$\|\Psi_\omega\|_{C^1} < 2\varepsilon_3 \quad (3.10)$$

and

$$-\Delta(\Phi_\omega + \Psi_\omega) = R^2(\Phi_\omega + \Psi_\omega). \quad (3.11)$$

Proof of the claim

Take $M \sim \sqrt{\log K} \sqrt{\log \frac{R}{\varepsilon_2}}$ and subdivide the M -cube centered at 0 in \mathbb{C}^K in cubes Q_α of size $\varepsilon_4 = \frac{\varepsilon_3}{R\sqrt{K}}$. Their number is $O((M\varepsilon_4^{-1})^{2K})$.

For each α , denote

$$A_\alpha = \{x \in \mathbb{T}^2; (c_k(x))_{1 \leq k \leq K} \in Q_\alpha\}$$

and

$$B_\alpha = \{\omega \in \Omega; (g_k(\omega))_{1 \leq k \leq K} \in Q_\alpha\}$$

with Ω the probability space on which Φ_ω is defined. According to Lemma 2, we can ensure that

$$||A_\alpha| - |B_\alpha|| < \varepsilon \text{ for each } \alpha.$$

Note that $|B_\alpha| > \delta(K, M, \varepsilon_4)$ and hence, for ε small enough, we may ensure

$$||A_\alpha| - |B_\alpha|| < \frac{\varepsilon_2}{2R^2}|B_\alpha|. \quad (3.12)$$

This permits to introduce subsets $A'_\alpha \subset A_\alpha, B'_\alpha \subset B_\alpha$, such that

$$|A'_\alpha| = |B'_\alpha| > \left(1 - \frac{\varepsilon_2}{2R^2}\right)|B_\alpha| \quad (3.13)$$

and a measure preserving map

$$\tau_\alpha : B'_\alpha \rightarrow A'_\alpha.$$

Define on $\cup B'_\alpha$

$$\Psi_\omega(y) = \frac{1}{\sqrt{K}} \sum_{k=1}^K [c_k(\tau_\alpha(\omega)) - g_k(\omega)] e(\zeta^{(k)} \cdot y) + \psi_{\tau_\alpha(\omega)} \quad (3.14)$$

and set

$$\Psi_\omega = 0 \text{ if } \omega \notin \cup B'_\alpha.$$

With this construction,

$$\begin{aligned} \int N_{\Phi_\omega + \Psi_\omega} d\omega &= \\ \sum_\alpha \int_{B'_\alpha} N_{\varphi_{\tau_\alpha(\omega)} + \psi_{\tau_\alpha(\omega)}} d\omega &+ \int_{(\cup B'_\alpha)^c} N_{\Phi_\omega} d\omega \\ &= \sum_\alpha \int_{A'_\alpha} N_{\varphi_x + \psi_x} dx + O(R^2 |(\cup B'_\alpha)^c|) \end{aligned}$$

$$= \int_{\mathbb{T}^2} N_{\varphi_x + \psi_x} dx + O(R^2 |(\cup A'_\alpha)^c|) + O(R^2 |(\cup B'_\alpha)^c|) \quad (3.15)$$

where we have used again that

$$N_{\varphi_x + \psi_x} < O(R^2) \text{ and } N_{\Phi_\omega} < O(R^2).$$

Next

$$\begin{aligned} |(\cup B'_\alpha)^c| &= \sum_\alpha |B_\alpha \setminus B'_\alpha| \\ &\quad + |\{\omega; \max_{1 \leq k \leq K} |g_k(\omega)| > M\}| \\ &\stackrel{(3.13)}{<} \frac{\varepsilon_2}{R^2} \sum |B_\alpha| + \frac{\varepsilon_2}{R^2} < 2 \frac{\varepsilon_2}{R^2} \end{aligned}$$

and

$$|(\cup A'_\alpha)^c| = \sum_\alpha |A_\alpha \setminus A'_\alpha| + |\{x \in \mathbb{T}^2; \max_{1 \leq k \leq K} |c_k(x)| > M\}|.$$

Again by (3.13)

$$\sum_\alpha |A_\alpha \setminus A'_\alpha| \leq \left(1 + \frac{\varepsilon_2}{2R^2}\right) \sum |B_\alpha| - \sum |B'_\alpha| < \frac{\varepsilon_2}{R^2}.$$

From Lemma 2

$$\begin{aligned} |\{x \in \mathbb{T}^2; \max_{1 \leq k \leq K} |c_k(x)| > M\}| &\leq \sum_{k=1}^K \text{mes}[|c_k| > M] \\ &< K(\varepsilon + \text{mes}[|g_k| > M]) < \frac{\varepsilon_2}{R^2}. \end{aligned}$$

Substituting in (3.15) gives

$$\int_{\mathbb{T}^2} N_{\Phi_\omega + \Psi_\omega} d\omega = \int_{\mathbb{T}^2} N_{\varphi_x + \psi_x} dx + O(\varepsilon_2). \quad (3.16)$$

Finally, note that on B'_α , by (3.14) and choice of ε_4

$$\|\Psi_\omega\|_{C^1} \leq R\sqrt{K}\varepsilon_4 + \|\psi_{\tau_\alpha(\omega)}\|_{C^1} < 2\varepsilon_3.$$

Also, since either $\Psi_\omega = 0$ or $\Phi_\omega + \Psi_\omega = \varphi_x + \psi_x$ for some x , it follows that $-\Delta(\Phi_\omega + \Psi_\omega) = R^2(\Phi_\omega + \Psi_\omega)$. This proves the claim. \square

At this stage, we are reduced to study the expected number of nodal domains in $]-\frac{1}{2}, \frac{1}{2}[\times]-\frac{1}{2}, \frac{1}{2}[$ of the perturbed Gaussian vector Φ_ω .

We make use of the work of Nazarov–Sodin and more specifically, several results from [S].

First there is the stability issue. Considering the random Gaussian function Φ_ω given by (3.8), clearly

$$\mathbb{E}_\omega[\|\Phi_\omega\|_{C^2}] < O(R^2). \quad (3.17)$$

Invoking Lemma 5 from [S], which is based on the independence of Φ_ω and $\nabla\Phi_\omega$, we get some $\beta = \beta(R, \varepsilon_2) > 0$ such that

$$\min_{x \in [-\frac{1}{2}, \frac{1}{2}]^2} \max(|\Phi_\omega(x)|, |\nabla\Phi_\omega(x)|) > \beta \quad (3.18)$$

for all ω outside a set of measure less than $\frac{\varepsilon_2}{R^2}$, hence contributing to

$$\mathbb{E}[N_{\Phi_\omega + \Psi_\omega}]$$

for at most $O(\varepsilon_2)$.

Property (3.18) is crucial to derive a stability property for the number of nodal domains under perturbation (see [S], Lemma 6). Recall that the perturbation Ψ_ω satisfies $\|\Psi_\omega\|_{C^1} < \varepsilon_3$. Taking

$$\varepsilon_3 = \beta(R, \varepsilon_2) \frac{1}{10R}. \quad (3.19)$$

Lemma 7 from [S] applied with $f = \Phi_\omega, g = \Psi_\omega$ and $\alpha = 2\varepsilon_3$ implies in particular the following

$N_{\Phi_\omega + \Psi_\omega} \geq$ number of components of $Z(\Phi_\omega)$ contained in the square

$$Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \text{ and at distance at least } \frac{1}{2R} > \frac{2\alpha}{\beta} \text{ from } \partial Q. \quad (3.20)$$

Note that since $\|\Psi_\omega\|_{C^1} < \varepsilon_3$, (3.18) also implies that

$$\min_{x \in Q} \max(|(\Phi_\omega + \Psi_\omega)(x)|, |\nabla(\Phi_\omega + \Psi_\omega)(x)|) > \frac{\beta}{2}. \quad (3.21)$$

Another application of [S], Lemma 7 taking $f = \Phi_\omega + \Psi_\omega, g = -\Psi_\omega$ yields conversely that

$$\text{Number of components of } Z(\Phi_\omega) \text{ contained in } \left[-\frac{1}{2} - \frac{1}{2R}, \frac{1}{2} + \frac{1}{2R}\right]^2 \geq N_{\Phi_\omega + \Psi_\omega}. \quad (3.22)$$

It follows from the two-sided inequalities (3.20), (3.22) that

$$\begin{aligned} & |\mathbb{E}[N_{\Phi_\omega}] - \mathbb{E}[N_{\Phi_\omega + \Psi_\omega}]| < \\ & \mathbb{E}\left[\# \text{ components } C \text{ of } Z(\phi_\omega) \text{ contained in } \left[-\frac{1}{2} - \frac{1}{2R}, \frac{1}{2} + \frac{1}{2R}\right]^2 \text{ for which} \right. \\ & \left. \text{dist}(C, \partial Q) < \frac{1}{R}\right] \\ & + O(\varepsilon_2). \end{aligned} \quad (3.23)$$

Recalling (3.8) and $\zeta_k = \frac{R}{\lambda}\xi_k$ with $\xi_k \in \mathbb{Z}^2, |\xi_k| = \lambda$, we have

$$N_{\Phi_\omega} = N\left(0, \frac{R}{\lambda}, f_\omega\right)$$

defining

$$f_\omega(x) = \frac{1}{\sqrt{K}} \sum_{k=1}^K g_k(\omega) e(\xi_k \cdot x) \quad (3.24)$$

and $N(0, \frac{R}{\lambda}; f_\omega)$ the number of components of $Z(f_\omega)$ contained in $Q_R =]-\frac{R}{2\lambda}, \frac{R}{2\lambda}[\times]-\frac{R}{2\lambda}, \frac{R}{2\lambda}[$. Thus f_ω is a Gaussian random eigenfunction of \mathbb{T}^2 of eigenvalue E .

The first term in (3.23) accounts for the number of components C of f_ω contained in Q_{R+1} and such that $\text{dist}(C, \partial Q_R) < \frac{1}{\lambda}$. Each of these components has area at least $O(\frac{1}{\lambda^2})$ and it follows from the Kac-Rice formula that

$$\mathbb{E}_\omega[\text{length}(Z_{f_\omega} \cap Q_{R+1})] < O\left(\frac{R^2}{\lambda}\right). \quad (3.25)$$

From these facts, one easily derives that

$$\mathbb{E}\left[\# \text{ components } C \text{ of } Z(f_\omega) \text{ s.t. } C \subset Q_{R+1}, \text{dist}(C, \partial Q_R) < \frac{1}{\lambda}\right] < O(R^{\frac{3}{2}} \log R) \quad (3.26)$$

(we first exclude those components C of size at least $\log R \cdot \frac{\sqrt{R}}{\lambda}$, assuming $\text{length}(Z_{f_\omega} \cap Q_{R+1}) < \log R \cdot \frac{R^2}{\lambda}$ and then exploit the area lower bound of each component).

Hence we proved that

$$\mathbb{E}[N_{\Phi_\omega + \Psi_\omega}] = \mathbb{E}\left[N\left(0, \frac{R}{\lambda}, f_\omega\right)\right] + O(\varepsilon_2 + R^{3/2} \log R). \quad (3.27)$$

The expectation of $N(0, \frac{R}{\lambda}, f_\omega)$ in the $\lim_{R \rightarrow \infty} \lim_{\lambda \rightarrow \infty}$ is given by Theorem 5 in [S] and we get in our situation

$$R^2 \cdot \nu(\rho) \quad (3.28)$$

where $\nu(\rho)$ is the constant given by [S], Theorem 1 associated to the measure ρ , which is the limiting spectral measure of our sequence (3.24), in the sense of [S]. Thus one considers the spectral measure ρ_λ associated to (3.24) defined by

$$\widehat{\rho_\lambda}(u - v) = \mathbb{E}\left[f_\omega\left(\frac{u}{\lambda}\right) \cdot f_\omega\left(\frac{v}{\lambda}\right)\right] = \frac{1}{K} \sum_{k=1}^K e\left(\frac{\xi_k}{\lambda} \cdot (u - v)\right). \quad (3.29)$$

Hence

$$\rho_\lambda = \frac{1}{K} \sum_{k=1}^K \rho_{\xi_k \lambda^{-1}}$$

where δ_z stands for the Dirac measure at $z \in \mathbb{R}^2$, $|z| = 1$.

Since, by assumption (D) and the construction in Section 2, the measures ρ_λ become equidistributed for $\lambda, K \rightarrow \infty$, the limiting measure ρ is the normalized Lebesgue measure on the unit circle and $\bar{\nu}$ is the constant associated to the RWM discussed in the Introduction; i.e. $\sigma = 4\pi\bar{\nu}$. Recall (3.16), (3.27) and take say $\varepsilon_2 = R^{-\frac{1}{10}}$. From the preceding, we obtain

$$\int_{\mathbb{T}^2} N_{\varphi_x + \psi_x} dx = (\bar{\nu} + o(1)) R^2$$

and

$$\int_{\mathbb{T}^2} N_{F_x} dx = (\bar{\nu} + o(1)) R^2. \quad (3.30)$$

In view of (3.1), (3.2), we obtain finally from the choice of ε_2 , that

$$N_f = (\bar{\nu} + o(1))\lambda^2.$$

Recapitulating the preceding, the order in which the various parameters are chosen is

$$R, \varepsilon_2, \beta, \varepsilon_3, K, M, \varepsilon_4, \varepsilon, \varepsilon_1, B(K, \varepsilon).$$

We proved the following

Proposition 1. *Assume E taken in a sequence such that (D) holds for $E \rightarrow \infty$ and also, for some fixed $\gamma < \frac{1}{2}$, condition $I(\gamma, B(E))$ with $B(E) \xrightarrow{E \rightarrow \infty} \infty$. Let*

$$f_E = \sum_{\xi \in \mathcal{E}_E} e(x \cdot \xi)$$

or, more generally

$$f_E = \sum_{\xi \in \mathcal{E}_E} a_\xi e(x \cdot \xi) \text{ with } a_{-\xi} = \bar{a}_\xi, |a_\xi| = 1.$$

Then the number N_E of nodal domains of f_E satisfies

$$4\pi \frac{N_E}{E} \rightarrow \sigma$$

4. ARITHMETIC CONSIDERATIONS

We return to the assumptions (D) of equidistribution and (I) of independence. Recall that we assumed E of the form

$$E = \prod p_\alpha^{e_\alpha} \tag{4.1}$$

with p_α odd, $p_\alpha \equiv 1 \pmod{4}$.

Let $\pi_\alpha = |\pi_\alpha| e^{i\theta_\alpha}$ and write $\xi = \lambda e^{i\psi_\xi}$ for $\xi \in \mathcal{E} = \mathcal{E}_E$, according to (1.6), (1.10).

We start with a statistical discussion, considering a ‘typical’ integer E of the above form.

A quantitative form of the required angular equidistribution is established in [E-H] (see Theorem 1).

Lemma 3. *Given $\varepsilon > 0$, for almost all integers E considered above, one has a discrepancy bound*

$$\Delta(E) < |\mathcal{E}_E| (\log E)^{-\kappa + \varepsilon} \text{ where } \kappa = \frac{1}{2} \log \frac{\pi}{2}. \tag{4.2}$$

Here $\Delta(E)$ is defined as

$$\max_{0 \leq \alpha < \beta < 2\pi} \left| \frac{\beta - \alpha}{2\pi} |\mathcal{E}_E| - [\#\{\xi \in \mathcal{E}; \psi_\xi \in [\alpha, \beta] \pmod{2\pi}] \right|. \tag{4.3}$$

The proof of this result depends on Kubilius’ evaluation of the number of Gaussian primes in a sector and bounds on multiplicative functions.

Let us also recall that, on average, an integer E that is sum of 2 squares has $\asymp \frac{1}{2} \log \log E$ prime factors, implying that $|\mathcal{E}_E| \sim \sqrt{\log E}$.

Next, we discuss the independence condition, again from a statistical perspective. The following statement follows from [B-B], Theorem 17 and Remark 15.

Lemma 4. *Given $\ell > 2$, for most integers E of type (4.1), the number of non-degenerate relations $\xi_1 + \dots + \xi_\ell = 0$ among elements of \mathcal{E}_E is at most $O(|\mathcal{E}_E|)$ for $E \rightarrow \infty$. More precisely, given any function φ , $\frac{\varphi(u)}{u} \xrightarrow{u \rightarrow \infty} \infty$, for most E , the number of non-degenerate solutions is bounded by $\varphi(|\mathcal{E}_E|)$.*

Obviously, this implies property $I(\gamma, B)$ for any given $\gamma > \frac{1}{3}$, for typical E taken large enough.

Note that [B-B], Theorem 17 follows from the following statement, which in some sense is stronger.

Denote

$$\Omega_{X,K} = \{E = \prod p_\alpha < X; p_\alpha \equiv 1 \pmod{4}, p_\alpha > K\}. \quad (4.4)$$

This set satisfies

$$|\Omega_{X,K}| \sim \frac{X}{\sqrt{\log X} \sqrt{\log K}}. \quad (4.5)$$

Theorem 14 in [B-B] asserts then that for fixed ℓ ,

$$\lim_{K \rightarrow \infty} \lim_{X \rightarrow \infty} \frac{1}{|\Omega_{X,K}|} |\{E \in \Omega_{X,K}; \mathcal{E}_E \text{ admits a nondegenerate relation of length } \ell\}| = 0. \quad (4.6)$$

Lemmas 3 and 4 are clearly addressing the assumptions from our theorem in the statistical sense. Thus we can state

Theorem 2. *The conclusion from Proposition 1 holds for almost all $E \rightarrow \infty$ of the form (1.4).*

Our next goal is a deterministic implementation. We start with the independence assumption. In certain cases, the desired information is provided by the deep work of Evertse-Schlickewei-Schmidt on additive relations in multiplicative subgroups of \mathbb{C}^* of bounded rank [E-S-S], which in turn depends on the subspace theorem. The result of [E-S-S] states that any unit equation

$$a_1 g_1 + \dots + a_\ell g_\ell = 1 \quad (4.7)$$

with g_1, \dots, g_ℓ taken in a multiplicative group G over \mathbb{C} of \mathbb{Q} -rank r , has at most $\exp(c(\ell)(r+1))$ non-degenerate solutions. Here $c(\ell)$ may be taken

$$c(\ell) = (4\ell)^{3\ell}. \quad (4.8)$$

An immediate consequence is the following

Lemma 5. *Let $E = \prod_{\alpha=1}^r p_{\alpha}^{e_{\alpha}}$ be as above. Then the number of non-degenerate relations (2.12) among elements from \mathcal{E}_E is bounded by*

$$\exp(c(\ell-1)(2r+1))|\mathcal{E}_E|. \quad (4.9)$$

While estimate (4.9) does not suffice in general to conclude a condition $I(\gamma, B)$, it does suffice provided $r = o(\log |\mathcal{E}_E|)$, i.e., recalling (1.7)

$$\frac{1}{r} \sum_{\alpha=1}^r \log e_{\alpha} \rightarrow \infty. \quad (4.10)$$

This is in particular the case if we fix the prime factors p_1, \dots, p_r of E and take their exponents e_{α} large enough.

Remark 1. It has been suggested that the true upper bound for the number of non-degenerate solutions of (4.7) may be subexponential in the rank, possibly bounded by

$$\exp(c(\ell)r^{\beta(\ell)}) \text{ for some } \beta(\ell) < 1. \quad (4.11)$$

If this were true, $I(\gamma, B)$ would hold with any $\gamma > \frac{1}{3}$, for all sufficiently large E .

Remark 2. If we fix $p_1, \dots, p_r \equiv 1 \pmod{4}$ and let $E = p_1^{e_1} \cdots p_r^{e_r}$ with $\max e_{\alpha} \rightarrow \infty$, condition (I) is certainly satisfied. Condition (D) amounts by (1.10) to equidistribution of the angular set

$$\left\{ 2 \sum_{\alpha=1}^r j_{\alpha} \theta_{\alpha}; 0 \leq j_{\alpha} \leq e_{\alpha} \right\} \pmod{1}. \quad (4.12)$$

Hence $\theta_{\alpha} \notin 2\pi\mathbb{Q}$. This is the case, since otherwise $\cos 2b\theta_{\alpha} = 1$ for some $b \in \mathbb{Z}_+$, implying that $\cos 2\theta_{\alpha}$ is an algebraic integer. But since $tg\theta_{\alpha} = \frac{\xi_2}{\xi_1} \in \mathbb{Q}$, $\cos 2\theta_{\alpha} \in \mathbb{Q}$ and therefore $\cos 2\theta_{\alpha} \in \mathbb{Z}$, $\theta_{\alpha} \in \frac{\pi}{4}\mathbb{Z}$ (contradiction).

Hence

Theorem 3. *Assume given $p_1, \dots, p_r \equiv 1 \pmod{4}$. Then the conclusion from Proposition 1 holds when E ranges in the set*

$$\{p_1^{e_1} \cdots p_r^{e_r}, e_1, \dots, e_r \in \mathbb{Z}_+\}.$$

Remark 3. We may also state the following property, which results from (4.7), (4.8) and an easy adaptation of the proof of Lemma 2.

Theorem 4. *Fix r and let E range in a sequence of integers of the form*

$$E = \prod_{\alpha=1}^r p_{\alpha}^{e_{\alpha}} \quad (p_{\alpha} \equiv 1 \pmod{4}).$$

For each E , let

$$f_E = \sum_{\xi \in \mathcal{E}_E} a_{\xi} e(x \cdot \xi) \quad (a_{-\xi} = \bar{a}_{\xi}) \quad (4.13)$$

$$\sum |a_\xi|^2 = 1$$

be arbitrarily chosen, subject to the assumption that the probability measures

$$\rho_E = \sum_{\xi \in \mathcal{E}_E} |a_\xi|^2 \delta_{\lambda^{-1}\xi} \quad (\lambda^2 = E) \quad (4.14)$$

on the unit circle, converge weak* to the normalized Lebesgue measure on S^1 for $E \rightarrow \infty$.

Denoting N_E the number of nodal domains of E , we have that

$$\frac{N_E}{E} \rightarrow \bar{\nu} \text{ for } E \rightarrow \infty. \quad (4.15)$$

5. FURTHER COMMENTS

An alternative approach would consist in considering ‘jets’ of eigenfunctions. Thus given an eigenfunction f_E of \mathbb{T}^2 , introduce at each point $x \in \mathbb{T}^2$ the scaled function

$$\varphi(y) = \varphi_x(y) = f\left(x + \frac{y}{\lambda}\right). \quad (5.1)$$

The function φ may be ε -approximated on $[|y| < R]$ by truncation of its Taylor expansion at order $B = B(R, \varepsilon)$, leading to a jet

$$J_x = \{D^\alpha \varphi_x|_{y=0}\}_{|\alpha| < B}. \quad (5.2)$$

Consider J_x as a random vector in $x \in \mathbb{T}^2$. Under assumptions (D) and (I), one may then show that the distribution of $(J_x)_{x \in \mathbb{T}^2}$ is approximatively the same as for the Gaussian random function with circular spectral measure and derive from this that $\frac{N_E}{E} \rightarrow \bar{\nu}$. This approach has the advantage of at least conceptually generalizing to real analytic compact manifolds M . Following [S], Section 2, one considers a map (assuming $\dim M = 2$)

$$\Phi_x = \exp_x \circ I_x : \mathbb{R}^2 \rightarrow M, \Phi_x(0) = x$$

with $\exp_x : T_x M \rightarrow X$ the exponential map and $I_x : \mathbb{R}^2 \rightarrow T_x(M)$ a linear Euclidean isometry. The function φ_x is then defined by

$$\varphi_x(y) = f(\Phi_x(\lambda^{-1}y)). \quad (5.3)$$

But we preferred to follow the procedure adopted earlier because it is more explicit and, in any case, we do not have examples at this point, other than the flat torus, where the RWM may be implemented deterministically.

This discussion may however be of interest in the (arithmetic) hyperbolic case. (See [G-R-S] for some remarkable new results on nodal domains in this setting).

Basically, the required behavior of the $(J_x)_{x \in M}$ may here in some sense be seen as a far generalization of the Gaussian distribution conjecture of the eigenfunctions.

REFERENCES

- [B] J. Bourgain, On Pleijel's nodal domain theorem, preprint 2012, to appear in IMRN.
- [Ber] M. Berry, Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature, J. Phys. A: Math. Gen. 35 (2002), 3025–3038.
- [B-B] E. Bombieri, J. Bourgain, A problem on sums of two squares, preprint 2012 (submitted to IMRN).
- [B-S1] E. Bogomolny, C. Schmit, Percolation model for nodal domains of chaotic wave functions, Phys. Rev. Letters, 88 (2002), 114102.
- [B-S2] E. Bogomolny, C. Schmit, Random wave functions and percolation J. Phys.A 40 (2007), 14033–14043,
- [E-S-S] J. Evertse, H. Schlickewei, W. Schmidt, Linear equations with variables which lie in a multiplicative group, Annals of Math (2), 155 (2002), 807–836.
- [E-H] P. Erdős, R. Hall, On the angular distribution of Gaussian integers with fixed norm, Discrete Math 200 (1999), 87–94.
- [F-K-W] L. Fainsilber, P. Kurlberg, B. Wennberg, Lattice points on circles and discrete velocity models for the Boltzmann equation.
- [G-R-S] A. Ghosh, A. Reznikov, P. Sarnak, Nodal domains of Maass forms, arXiv:1207.6625.
- [K] M. Krishnapur, Continuing nodal domains in random plane waves, preprint 2012,
- [L] H. Lewy, On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere, Comm. Part. Diff. Eqs 2 (1977), 1233–1244.
- [N-S] F. Nazarov, M. Sodin, On the number of nodal domains of random spherical harmonics, Amer. J. Math 131 (2009), 1337–1357.
- [S] M. Sodin, Lectures on random nodal portraits, preprint.
- [St] A. Stern, Bemerkungen über asymptotisches Verhalten von Eigenwerken und Eigenfunktionen, Dissertation, Göttingen 1925.

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